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LETTER TO THE EDITOR

Exponential potential and mass dependence of wavefunctions

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Abstract. We derive exact s-wave orthonormal eigenfunctions for an exponential potential in the non-relativistic case and show that the condition $G(r) \geqslant 0$ for $r \geqslant 0$ where

$$G(r) = \frac{\partial}{\partial m} \int_0^r (u(r))^2 \, \mathrm{d}r$$

does not hold in general.

1. Introduction

After the discoveries of bound states like the ψ/J and γ particles many workers (for a brief review see Quigg (1978)) have attempted to describe the spectrum of the new particles with the help of various potential models. Mass dependence of Schrödinger wavefunctions is of considerable interest in these studies. Recently Leung and Rosner (1979) studied mass dependence of wavefunctions in the non-relativistic case for some types of potentials. For a power law potential of the type $V(r) = r^{\epsilon}$ (and also for the logarithmic potential) mass dependence of the wavefunctions and energies (see Cocconi (1978)) can be found out easily by using simple scaling arguments. However, for a potential which is a transcendental function of r, such scaling arguments do not hold universally. Hence mass dependence of wavefunctions for some well known monotonic potentials like $\mathrm{e}^{-r/a}$ (the exponential type), e^{-kr}/r (Yukawan) or $A/\cos h^2 \alpha r$ (modified Pöschteller) cannot be determined in a trivial manner.

In this Letter we have derived exact s-wave orthogonal eigenfunctions for an exponential potential in the non-relativistic case and have shown that the condition $G(r) \geqslant 0$ for $r \geqslant 0$, where

$$G(r) = \frac{1}{2} \frac{\partial}{\partial m} \int_0^r (u(r))^2 \, \mathrm{d}r,$$

does not hold in general.

2. Solution of Schrödinger equation

The exact s-wave solutions of the Schrödinger equation for an exponential potential are well known. However, for the sake of completeness, we present here the essential

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steps. As usual we write the radial part of the s-wave Schrödinger wavefunction $\psi(r)$ as

$$\psi(r) = u(r)/r \tag{1}$$

where the radial wavefunction $u(r)$ satisfies the differential equation (taking $\hbar = 1$)

$$u'' + 2\mu(E - V(r))u = 0. \tag{2}$$

If we consider the equation for a bound state then μ is the reduced mass and $\mu = m/2$, where m is the mass of any of the constituent particles assumed to be of equal masses. If we take

$$V(r) = A\ \mathrm{e}^{-kr} \tag{3}$$

then (2) can be written as

$$v''(z) + (4m/k^2)(E - A\ \mathrm{e}^{2z})v(z) = 0 \tag{4}$$

where $u(r) = v(z)$ and $z = -kr/2$. The solution of (4) is given by

$$u(r) = a_0 \mathrm{J}_\nu(x\ \mathrm{e}^{-kr/2}) \tag{5}$$

where

$$\nu^2 = -4mE/k^2, \qquad \lambda^2 = -4Am/k^2 \tag{6}$$

and a_0 is a normalisation constant to be determined from the relation

$$\int_0^\infty (u(r))^2\ \mathrm{d}r = 1. \tag{7}$$

Eigenvalues are obtained from the boundary condition

$$\mathrm{J}_\nu(\lambda) = 0. \tag{8}$$

Now

$$\int_0^\infty \mathrm{J}_\nu(\lambda\ \mathrm{e}^{-kr/2})\mathrm{J}_\mu(\lambda\ \mathrm{e}^{-kr/2})\ \mathrm{d}r = \frac{2}{k}\int_0^1 \mathrm{J}_\mu(\lambda t)\mathrm{J}_\nu(\lambda t)t^{-1}\ \mathrm{d}t. \tag{9}$$

The RHS of (9) can be evaluated by using the explicit expressions for integrals like (see Luke (1962) equations (3)–(6))

$$\int^z z^{-1}\mathrm{J}_\mu(\lambda z)\mathrm{J}_\nu(\lambda z)\ \mathrm{d}z.$$

After some straightforward calculation, we see that the RHS of (9) is equal to zero if $\mu \neq \nu$, and for $\mu = \nu$

$$\int_0^1 (\mathrm{J}_\nu(\lambda t))^2 t^{-1}\ \mathrm{d}t = \frac{\lambda}{2\nu}\left(\mathrm{J}_{\nu+1}\frac{\partial \mathrm{J}_\nu(\lambda)}{\partial\nu}\right) \tag{10}$$

where we have also used the result (8).

Hence orthonormal eigenfunctions for an exponential potential are given by

$$u_n = a_n(\nu)\mathrm{J}_{\nu_n}(\lambda\ \mathrm{e}^{-kr/2}) \tag{11}$$

where ν_n is the value of ν corresponding to the nth zero of $\mathrm{J}_\nu(\lambda)$ for fixed λ. a_n is given by

$$a_n(\nu) = \left(\frac{k\nu}{\lambda}\right)^{1/2}\left(\mathrm{J}_{\nu+1}(\lambda)\frac{\partial \mathrm{J}_\nu}{\partial\nu}(\lambda)\right)^{-1/2} \tag{12}$$

where

$$\frac{\partial}{\partial\nu}J_\nu(\lambda)\equiv\frac{\partial}{\partial\nu}J_\nu(z)\bigg|_{z=\lambda}. \tag{13}$$

The normalisation constant given by (12) is, to the best of the author's knowledge, not given elsewhere. As is well known, in a bound state potential model, the decay width Γ is proportional to $(\psi(0))^2$ (see Quigg and Rosner (1978)).

To derive $|\psi(0)|^2$ we use

$$|\psi(0)|^2=(m/4\pi)\langle dV/dr\rangle. \tag{14}$$

Now

$$\begin{aligned}\left\langle\frac{dV}{dr}\right\rangle &= -ka_0^2\int_0^\infty (J_\nu(\lambda\ e^{-kr/2}))^2A\ e^{-kr}\ dr\\ &=2a_0^2A\int_0^1 (J_\nu(\lambda t))^2t\ dt\\ &=-a_0^2AJ_{\nu-1}(\lambda)J_{\nu+1}(\lambda)\end{aligned} \tag{15}$$

where we have used (Luke 1962, equations (3)–(6)) the explicit expression for

$$\int^z tJ_\nu(kt)J_\nu(Rt)\ dt$$

and equation (8).

Using the properties of Bessel functions,

$$\langle dV/dr\rangle=Aa_0^2(J_\nu'(\lambda))^2. \tag{16}$$

Hence

$$|\psi(0)|^2=(a_0^2Am/4\pi)(J_\nu'(\lambda))^2. \tag{17}$$

Let us now calculate $P(r)$ defined by

$$P(r)=\int_0^r (u(r))^2\ dr$$

$$=\frac{2}{k}\int_{e^{-kr/2}}^1 a_0^2(J_\nu(\lambda t))^2t^{-1}\ dt \tag{18}$$

$$=\frac{2}{R}\int_0^1 a_0^2(J_\nu(\lambda t))^2t^{-1}\ dt-\frac{2}{R}a_0^2\int_0^{e^{-kr/2}} (J_\nu(\lambda t))^2t^{-1}\ dt. \tag{19}$$

Using (10) and (12)

$$P(r)=1-\frac{e^{-kr/2}J_{\nu+1}(\lambda\ e^{-kr/2})\partial J_\nu(\lambda\ e^{-kr/2})/\partial\nu}{J_{\nu+1}(\lambda)\partial J_\nu(\lambda)/\partial\nu} \tag{20}$$

$$=1-f(m,r)\qquad\text{(say)} \tag{21}$$

where

$$f(m,r)=\frac{e^{-kr/2}J_{\nu+1}(\lambda\ e^{-kr/2})\partial J_\nu(\lambda\ e^{-kr/2})/\partial\nu}{J_{\nu+1}(\lambda)\partial J_\nu(\lambda)/\partial\nu}. \tag{22}$$

If $\partial P(r)/\partial m \geqslant 0$, then $P(r)$ is monotonically increasing, hence $f(m, r)$ must be a monotonically decreasing function of m. We show that this cannot be true except in the case when λ is the lowest zero (excluding $\lambda = 0$) of $J_\nu(\lambda)$. This follows from the interlace properties of the zeros of Bessel functions. (For the following discussions, we assume that $\lambda > 0$, $\nu > 0$.) For a higher-order zero of $J_\nu(z)$, there is a value of $r > 0$ for which

$$J_{\nu+1}(\lambda\ e^{-kr/2}) = 0. \tag{23}$$

Hence $f(m, r) = 0$.

For example if

$$m = j_{\nu,n}^2 k^2/4A \tag{24}$$

then

$$J_{\nu+1}(\lambda\ e^{-kr/2}) = 0 \tag{25}$$

for

$$kr = 2\ln j_{\nu,n}/j_{\nu+1,n-1} \tag{26}$$

where $j_{\nu,n}$ is the nth zero of $J_\nu(z)$. The above result follows from the 'interlace' properties of $j_{\nu,n}$,

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} \ldots < j_{\nu+1,n-1} < j_{\nu,n}. \tag{27}$$

3. Conclusion

The condition $G(r) \geqslant 0$, $0 \leqslant r < \infty$ is a quantitative statement that the bound particle 'falls deeper into the well' (as was pointed out by Leung and Rosner (1979)) as μ increases. As shown by us this condition fails to hold in case of an exponential potential. Our investigation was limited to the case $\nu > 0$, $\lambda > 0$. When both ν and λ become purely imaginary, i.e. $A > 0$, $E > 0$, the investigation becomes very much more difficult as little is known about the properties of complex roots of Bessel functions with complex arguments. But our results up to equation (22) hold good for any ν and λ.

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